Proof systems for the logics of bringing-it-about

Tiziano Dalmonte, Charles Grellois, and Nicola Olivetti [1]

Aix Marseille Univ, Université de Toulon, CNRS, LIS, Marseille, France

Abstract

The logic of Bringing-it-About was introduced by Elgesem to formalise the notions of agency and capability. It contains two families of modalities indexed by agents, the first one expressing what an agent brings about (does), and the second expressing what she *can* bring about (can do). We first introduce a new neighbourhood semantics, defined in terms of bi-neighbourhood models for this logic, which is more suited for countermodel construction than the semantics defined in the literature. We then introduce a hypersequent calculus for this logic, which leads to a decision procedure allowing for a practical countermodel extraction. We finally extend both the semantics and the calculus to a coalitional version of Elgesem logic proposed by Troquard.

Keywords: Logic of agency, logic of ability, coalition logic, sequent calculus, countermodel extraction, decision procedure.

1 Introduction

The logic of Bringing-It-About was originally proposed by Elgesem [5], and provides one possible formalisation of agents' actions in terms of their results: that an agent "does something" is interpreted as the fact that the agent brings about something, for instance "John does a bank transfer" is interpreted as "John does that the bank transfer is done". The logical system proposed by Elgesem contains two modalities indexed by agents $\mathbb{E}_i$ and $\mathbb{C}_i$ (this is not his original notation), the former expressing the agentive modality of bringing-it-about, and the latter expressing capability, roughly speaking $\mathbb{E}_{lucy}\, \mathit{BankTransfer}$ means that Lucy makes a bank transfer, whereas $\mathbb{C}_{lucy}\, \mathit{BankTransfer}$ means that Lucy *can* make a bank transfer. Elgesem's logic is then intended to capture the effect of the action "what is brought about" and the agency relation, abstracting away from any temporal and game-theoretic aspect. In this way it provides a terse formalism, that has become a standard, quite simpler than other formalisms such as STIT-logic [2,8]. Elgesem's logic is well-suited for expressing notions of *responsibility* and formalising notions of control, power, and delegation, for instance: "Sara prevents Lucy from making a bank transfer" will be captured just

[1] {tiziano.dalmonte,charles.grellois,nicola.olivetti}@lis-lab.fr. This work has been partially supported by the ANR project TICAMORE ANR-16-CE91-0002-01.

by $\mathbb{E}_{sara} \neg \mathbb{E}_{lucy}$ *BankTransfer*; moreover it can be easily combined with deontic modalities in order to express e.g. that an agent is *obliged* to do something and so on.

Elgesem proposed an axiomatisation of his logic and a (almost matching) semantics based on selection function models. Notice that the intended notion of capability is rather weak, the only characterising axioms are that (i) agency implies capability $\mathbb{E}_i A \to \mathbb{C}_i A$ and (ii) $\neg \mathbb{C}_i \top$, the latter expressing that an agent i is not capable of doing anything that is always true, whence also $\neg \mathbb{E}_i \top$: an agent cannot do anything that will happen anyway, no matter her own involvement and responsibility.

Elgesem's logic was further studied by Governatori and Rotolo [7], who proposed an alternative semantics in terms of neighbourhood models. In their semantics, models contain two neighbourhood functions corresponding to the two operators $\mathbb{E}_i$ and $\mathbb{C}_i$ assigning for each agent i the propositions (identified with their truth sets) that the agent i brings/can bring about. They also proved that Elgesem's semantics entails the validity of the further axiom $\neg \mathbb{C}_i \bot$ meaning that an agent cannot bring about something which is contradictory.

Elgesem's logic deals with actions of a single agent, who might be either a human individual, or an institution, or a group conceived as an indivisible entity. A natural extension of this logic is to handle groups or coalitions that act *jointly* to bring about an action. This has been proposed by Troquard [11] who has developed an extension of Elgesem logic to handle "coalitions": individuals may gather in coalitions to bring about a joint action. In a joint action, each participant must be involved, so that the logic rejects coalition monotonicity: $\mathbb{E}_g A \to \mathbb{E}_{g'} A$ whenever $g \subseteq g'$ is *not* considered as valid. Troquard provided a computational analysis of his logic and determined its complexity by providing a decision procedure for his logic, whence for Elgesem's.

While the semantics of Elgesem logic, as well as its coalitional extension are well-understood, its proof-theory is mainly unexplored: the only known proof system for this logic was proposed by Lellmann [9]. In particular, no proof system connecting the syntax and the semantics is known. By this we mean that there is no proof system so far that permits the construction of countermodels of non-valid formulas. Moreover, no proof system is known at all for the coalitional extension. In particular, the decision procedure developed by Troquard [11] computes a reduction of a question about validity in his coalition logic to a set of SAT problems. This is in the spirit of the approach of Vardi [12] and Giunchiglia et al. [6] for non-normal modal logics. But this algorithms based on SAT-reduction does not provide neither *derivations*, nor *countermodels*.

This is precisely the purpose of this work. We take our move by redefining the semantics of Elgesem logic: we consider bi-neighbourhood models, a variant of neighbourhood models defined in [7]. Like the models in [7], our models contain, for each agent i, two neighbourhood functions corresponding to the two operators $\mathbb{E}_i$ and $\mathbb{C}_i$. But contrary to the neighbourhood models of [7], these functions assign to each world a set of *pairs* of neighbourhoods (α, β).

Although it would be pretentious to suggest here a new semantics of actions, we can suggest some intuitive interpretations of the pairs of neighbourhoods (α, β): given a proposition A representing the result of an action of an agent i, the two components (α, β) of the pairs can be understood respectively as specifying independently a set of situations α enabling i to bring about A and β preventing i from doing A. An alternative interpretation is as follows: since A must be true in all worlds (situations) in α and false in all worlds in β, the former can also be thought as a set of *possible outcomes* of A and the latter as a set of *impossible outcomes* of A.[2] In this second interpretation, each pair (α, β) can also be thought of as expressing a *lower* and an *upper* approximation of propositions that the agent brings/can bring about given a proposition.

Note that a bi-neighbourhood model can be transformed into a standard neighbourhood model of [7], and conversely.

No matter its intuitive interpretation, the bi-neighbourhood semantics has a clear technical advantage as it makes easier to compute countermodels of non-valid formulas than the standard neighbourhood semantics, by avoiding the exact determination of the truth sets of formulas.

We next move to proof theory by proposing a hypersequent calculus. A hypersequent can be thought of as a disjunction of ordinary sequents. While the hypersequent structure is not needed to obtain a complete calculus (as witnessed by [9] itself), the use of hypersequents allows us to define a calculus with invertible rules, as a difference with the one in [9]. The main advantage is that from *one* failed hypersequent occurring as a leaf of *one* derivation tree, a countermodel can directly be extracted in the bi-neighbourhood semantic of the formula under verification. In this sense, our calculus provides not only a decision procedure for this logic, but also the first practical procedure to compute countermodels. Observe that it is not possible to compute directly countermodels by ordinary sequent calculi: because the rules are not invertible, the fact that *one* specific derivation fails, does not mean that the sequent is unprovable, so that in order to build a countermodel (for a non-valid formula), all possible derivations must be attempted and inspected. Another syntactic feature of our calculi is that hypersequents contain additional structural constructs, the blocks, which are necessary for countermodel construction, but also to capture the logic in a clean and modular way, reflecting its axiomatisation.

The hypersequent calculus has nonetheless good proof-theoretic properties, as it enjoys *a syntactic proof of cut elimination*, from which also follows its completeness with respect to the axiomatisation. We then turn to the coalitional version of Elgesem's logic proposed by Troquard [11]: we are able to extend both the bi-neighbourhood semantics and the calculus to this setting, needing only to add the rules for handling the empty coalition and coalition fusion. Our calculus then provides a decision procedure for Troquard coalitional logic, with derivations and countermodels.

[2] We are grateful to one reviewer for suggesting this latter interpretation.

$\mathrm{RE}_{\mathbb{E}}$	$\dfrac{A \leftrightarrow B}{\mathbb{E}_i A \leftrightarrow \mathbb{E}_i B}$	$\mathrm{RE}_{\mathbb{C}}$	$\dfrac{A \leftrightarrow B}{\mathbb{C}_i A \leftrightarrow \mathbb{C}_i B}$
$\mathrm{C}_{\mathbb{E}}$	$\mathbb{E}_i A \wedge \mathbb{E}_i B \rightarrow \mathbb{E}_i(A \wedge B)$	$\mathrm{Q}_{\mathbb{C}}$	$\neg\mathbb{C}_i\top$
$\mathrm{T}_{\mathbb{E}}$	$\mathbb{E}_i A \rightarrow A$	$\mathrm{P}_{\mathbb{C}}$	$\neg\mathbb{C}_i\bot$
$\mathrm{Int}_{\mathbb{EC}}$	$\mathbb{E}_i A \rightarrow \mathbb{C}_i A$		

Fig. 1. Modal axioms and rules of Elgesem's logic **ELG**.

2 Elgesem's logic and bi-neighbourhood semantics

In this section, we present Elgesem's agency and ability logic, which we denote by **ELG**. Then we define the bi-neighbourhood models for this logic.

Let $\mathcal{A} = \{a, b, c, ...\}$ be a set of agents. The logic **ELG** is defined on a propositional language $\mathcal{L}_{Elg}$ containing, for every $i \in \mathcal{A}$, two unary modalities $\mathbb{E}_i$ and $\mathbb{C}_i$, respectively of "agency" and "ability". The formulas of $\mathcal{L}_{Elg}$ are defined by the following grammar:

$$A := p \mid \bot \mid \top \mid \neg A \mid A \wedge B \mid A \vee B \mid A \rightarrow B \mid \mathbb{E}_i A \mid \mathbb{C}_i A,$$

where $\mathbb{E}_i A$ and $\mathbb{C}_i A$ are respectively read as "the agent i brings it about that A", and "the agent i is capable of realising A". The logic **ELG** is defined by extending classical propositional logic (formulated in language $\mathcal{L}_{Elg}$) with the modal axioms and rules in Fig. 1. [3]

Notice that $\neg\mathbb{E}_i\bot$ and $\neg\mathbb{E}_i\top$ are derivable in **ELG**. By contrast, the axioms C and T hold only for the modality $\mathbb{E}$, meaning respectively that if an agent realises two things, then she realises both, and that if A is brought about by some agent, then it is actually the case that A.

Semantic characterisations of the logic **ELG** are provided by Elgesem [5] in terms of selection function models and by Governatori and Rotolo [7] in terms of neighbourhood models, the latter having separate neighbourhood functions for the modalities $\mathbb{E}$ and $\mathbb{C}$. Here we propose an alternative semantics based on bi-neighbourhood models [4]. We explain the advantages of this alternative semantics just after its definition.

Definition 2.1 A *bi-neighbourhood model* for **ELG** is a tuple $\mathcal{M} = \langle \mathcal{W}, \mathcal{N}_i^{\mathbb{E}}, \mathcal{N}_i^{\mathbb{C}}, \mathcal{V} \rangle$, where $\mathcal{W}$ is a non-empty set, $\mathcal{V}$ is a valuation function, and for each agent i, $\mathcal{N}_i^{\mathbb{E}}$ and $\mathcal{N}_i^{\mathbb{C}}$ are two bi-neighbourhood functions $\mathcal{W} \longrightarrow \mathcal{P}(\mathcal{P}(\mathcal{W}) \times \mathcal{P}(\mathcal{W}))$ satisfying the following conditions:

($\mathrm{C}_{\mathbb{E}}$) If $(\alpha, \beta), (\gamma, \delta) \in \mathcal{N}_i^{\mathbb{E}}(w)$, then $(\alpha \cap \gamma, \beta \cup \delta) \in \mathcal{N}_i^{\mathbb{E}}(w)$.
($\mathrm{T}_{\mathbb{E}}$) If $(\alpha, \beta) \in \mathcal{N}_i^{\mathbb{E}}(w)$, then $w \in \alpha$.
($\mathrm{Q}_{\mathbb{C}}$) If $(\alpha, \beta) \in \mathcal{N}_i^{\mathbb{C}}(w)$, then $\beta \neq \emptyset$.
($\mathrm{P}_{\mathbb{C}}$) If $(\alpha, \beta) \in \mathcal{N}_i^{\mathbb{C}}(w)$, then $\alpha \neq \emptyset$.
($\mathrm{Int}_{\mathbb{EC}}$) $\mathcal{N}_i^{\mathbb{E}}(w) \subseteq \mathcal{N}_i^{\mathbb{C}}(w)$.

[3] A variant of Elgesem's logic not containing axiom $\mathrm{P}_{\mathbb{C}}$ is considered in [7,9]. All results presented in this work can be extended to this variant just by dropping the corresponding condition in the bi-neighbourhood semantics and the corresponding rule in the calculus.

The forcing relation $\Vdash$ is defined as usual for atomic formulas and boolean connectives, whereas for $\mathbb{E}$- and $\mathbb{C}$-formulas it is defined as follows:

$$\begin{array}{lll} \mathcal{M}, w \Vdash \mathbb{E}_i A & \text{iff} & \text{there is } (\alpha, \beta) \in \mathcal{N}_i^{\mathbb{E}}(w) \text{ s.t.} \\ & & \text{for all } v \in \alpha,\ \mathcal{M}, v \Vdash A, \text{ and for all } u \in \beta,\ \mathcal{M}, u \not\Vdash A. \\ \mathcal{M}, w \Vdash \mathbb{C}_i A & \text{iff} & \text{there is } (\alpha, \beta) \in \mathcal{N}_i^{\mathbb{C}}(w) \text{ s.t.} \\ & & \text{for all } v \in \alpha,\ \mathcal{M}, v \Vdash A, \text{ and for all } u \in \beta,\ \mathcal{M}, u \not\Vdash A. \end{array}$$

Notice that if we denote by $[\![A]\!]$ the set $\{v \mid \mathcal{M}, v \Vdash A\}$, i.e., the *truth set* of A, the above clauses can be rewritten as $\mathcal{M}, w \Vdash \mathbb{E}_i A$ if and only if there is $(\alpha, \beta) \in \mathcal{N}_i^{\mathbb{E}}(w)$ s.t. $\alpha \subseteq [\![A]\!]$ and $\beta \subseteq [\![\neg A]\!]$, and similarly for $\mathbb{C}$-formulas. As usual, we omit to specify the model $\mathcal{M}$ when it is clear from context, and then we simply write $w \Vdash A$.

The main reason for considering bi-neighbourhood semantics is that is offers a much easier and natural way to extract countermodels from failed proofs. To see this, in the standard neighbourhood semantics, to make w satisfy $\mathbb{E}_i A$, *exactly* the truth set of A must belong to $\mathcal{N}_i^{\mathbb{E}}(w)$, whereas in the bi-neighbourhood semantics it is sufficient to find a pair (α, β) such that $\alpha \subseteq [\![A]\!]$ and $\beta \subseteq [\![\neg A]\!]$. Observe that this condition can be rewritten as $\alpha \subseteq [\![A]\!] \subseteq \mathcal{W} \setminus \beta$: in this way the pair (α, β) can be thought of as specifying a lower and upper approximation of the truth set of A. The fact that the exact determination of truth sets is not needed in the bi-neighbourhood semantics makes countermodels extraction from failed proofs substantially easier than in the standard semantics: a failed proof only specifies "partial" information, from which one can directly compute bi-neighbourhood pairs, but not exact truth-sets. For this reason bi-neighbourhood semantics is more natural for direct countermodel extraction than the standard one.

As mentioned in the introduction, bi-neighbourhood semantics can also have some intuitive meaning in terms of agency, we have suggested two possible interpretations: a bi-neighbourhood pair can be interpreted as a specification of enabling and preventing conditions for the realisation of actions, or as a set of possible/impossible outcomes of an action. In both interpretations, the conditions ($\mathrm{P}_{\mathbb{C}}$) and ($\mathrm{Q}_{\mathbb{C}}$), i.e., $\alpha \neq \emptyset$ and $\beta \neq \emptyset$ have a natural meaning: the former imposes that an action must be enabled or possible (non-empty possible outcomes), so that a contradiction cannot be realised; the latter imposes that an action must be preventable (non-empty impossible outcomes), so that a tautology cannot be realised.

Notice also that, because of the validity of $\neg\mathbb{E}_i\top$ and of the axiom $\mathrm{T}_{\mathbb{E}}$, formulas of the form $\mathbb{E}_i A$ are never valid in models for **ELG**, this is the semantic counterpart of the idea that actions can be always prevented.

Theorem 2.2 (Characterisation) *A is derivable in* **ELG** *if and only if it is valid in all bi-neighbourhood models for* **ELG**.

Proof. The proof of soundness is easy and amounts to showing that all axioms are valid and all rules are validity-preserving. Completeness can be proved by the canonical model construction as it is done in [4] for classical non-normal modal logics. Let us call **ELG**-maximal any set Φ of formulas of $\mathcal{L}_{Elg}$ such that $\Phi \nvdash_{\mathbf{ELG}} \bot$ and if $A \notin \Phi$, then $\Phi \cup \{A\} \vdash_{\mathbf{ELG}} \bot$. We denote by $\uparrow A$ the class of **ELG**-maximal sets containing A, and we define the canonical model for **ELG** as the tuple $\langle \mathcal{W}, \mathcal{N}_i^{\mathbb{E}}, \mathcal{N}_i^{\mathbb{C}}, \mathcal{V} \rangle$, where $\mathcal{W}$ is the class of maximal sets, $V(p) = \{\Phi \in \mathcal{W} \mid p \in \Phi\}$, and for every $i \in \mathcal{A}$ and $\mathbb{X} \in \{\mathbb{E}, \mathbb{C}\}$, $\mathcal{N}_i^{\mathbb{X}}(\Phi) = \{(\uparrow A, \mathcal{W} \setminus \uparrow A) \mid \mathbb{X}_i A \in \Phi\}$. We can prove that $\Phi \Vdash A$ if and only if $A \in \Phi$, (truth lemma, cf. [4]) and that the canonical model is a bi-neighbourhood model for **ELG**. We show as an example that it satisfies the conditions ($\mathrm{Q}_{\mathbb{C}}$) and ($\mathrm{Int}_{\mathbb{EC}}$): ($\mathrm{Q}_{\mathbb{C}}$) Assume $(\uparrow A, \mathcal{W} \setminus \uparrow A) \in \mathcal{N}_i^{\mathbb{C}}(\Phi)$. Then there is $\mathbb{C}_i B \in \Phi$ such that $\uparrow B = \uparrow A$, whence $\vdash B \leftrightarrow A$. If $\uparrow A = \mathcal{W}$, then $\vdash A \leftrightarrow \top$. Thus by $\mathrm{RE}_{\mathbb{C}}$, $\vdash \mathbb{C}_i B \leftrightarrow \mathbb{C}_i \top$, and since Φ is closed under derivation, $\mathbb{C}_i \top \in \Phi$, against the fact that $\neg \mathbb{C}_i \top \in \Phi$ and Φ is **ELG**-consistent. Therefore $\uparrow A \neq \mathcal{W}$, that is $\mathcal{W} \setminus \uparrow A \neq \emptyset$. ($\mathrm{Int}_{\mathbb{EC}}$) Assume $(\alpha, \beta) \in \mathcal{N}_i^{\mathbb{E}}(\Phi)$. Then there is $\mathbb{E}_i A \in \Phi$ such that $\alpha = \uparrow A$ and $\beta = \mathcal{W} \setminus \uparrow A$. Since $\mathbb{E}_i A \to \mathbb{C}_i A \in \Phi$ and Φ is closed under derivation, $\mathbb{C}_i A \in \Phi$. Thus $(\uparrow A, \mathcal{W} \setminus \uparrow A) = (\alpha, \beta) \in \mathcal{N}_i^{\mathbb{C}}(\Phi)$. □

Similarly to the transformation described in [3,4], a bi-neighbourhood model for **ELG** can be transformed into a neighbourhood model for it as follows (the proof is easy by induction on A):

Proposition 2.3 (Model transformation) *Let $\mathcal{M}_{bi} = \langle \mathcal{W}, \mathcal{N}_{bi}, \mathcal{V} \rangle$ be a bi-neighbourhood model for* **ELG***, and $\mathcal{M}_n = \langle \mathcal{W}, \mathcal{N}_n, \mathcal{V} \rangle$ be the neighbourhood model defined by taking the same $\mathcal{W}$ and $\mathcal{V}$ and, for all $w \in \mathcal{W}$,*

$$\mathcal{N}_n(w) = \{\gamma \subseteq \mathcal{W} \mid \text{there is } (\alpha, \beta) \in \mathcal{N}_b(w) \text{ such that } \alpha \subseteq \gamma \subseteq \mathcal{W} \setminus \beta\}.$$

Then, for every $A \in \mathcal{L}_{Elg}$ and every $w \in \mathcal{W}$, $\mathcal{M}_n, w \Vdash A$ if and only if $\mathcal{M}_{bi}, w \Vdash A$.

As the above transformation shows, bi-neighbourhood models have in general smaller functions than their equivalent neighbourhood models. The reason is that every bi-neighbourhood pair (α, β) – whose elements can be thought of as lower and upper bounds of neighbourhoods – might validate more than one modal formula.

3 Hypersequent calculus

We now focus on proof theory. To our knowledge, the only proof-theoretic investigation of Elgesem's logic is carried on in [9], where a cut-free sequent calculus is defined. That calculus provides a decision procedure for Elgesem's logic, but has no link with the semantics.

We propose here a hypersequent calculus (see [1]) for Elgesem's logic, in the same style of calculi for basic non-normal modal logics presented in [3]. A hypersequent can be loosely interpreted as a disjunction of sequents. The hypersequents considered in this article rely on an additional structure, called *blocks*. A block is used to collect $\mathbb{E}$- and $\mathbb{C}$-formulas: more precisely it represents a conjunction of formulas under the scope of the *same* $\mathbb{E}$ or $\mathbb{C}$. Since neither $\mathbb{E}$, nor $\mathbb{C}$ distribute over conjunction, blocks are not an abbreviation, they are a proper structural construct, and specific structural rules of the calculus handle them. Blocks within hypersequents are primarily needed for building countermodels of non-derivable formulas: as we will see, they are used to define bi-neighbourhood pairs. Blocks also have two other advantages: by using blocks we can encode in a clean (close to the axiomatisation) and analytic way the relation between the modalities $\mathbb{E}$ and $\mathbb{C}$; in addition the rules governing the modalities $\mathbb{E}$ and $\mathbb{C}$ are independent one of the other, so that the two $\mathbb{E}$ and $\mathbb{C}$-fragments are *separated*, and the interaction between the two modalities is captured just by a structural rule on blocks. We consider the following definitions:

Definition 3.1 (Block, sequent, hypersequent) A *block* is a structure $\langle\Sigma\rangle_i^{\mathbb{E}}$ or $\langle\Sigma\rangle_i^{\mathbb{C}}$, where i is an agent, and Σ is a multiset of formulas of $\mathcal{L}_{Elg}$. A *sequent* is a pair $\Gamma \Rightarrow \Delta$, where Γ is a multiset of formulas and blocks, and Δ is a multiset of formulas. We sometimes consider $\mathsf{set}(\Gamma)$, the *support* of a multiset Γ, i.e., the set of its elements disregarding multiplicities. A *hypersequent* is a multiset $S_1 \mid ... \mid S_n$, where $S_1, ..., S_n$ are sequents. $S_1, ..., S_n$ are called the *components* of the hypersequent.

Definition 3.2 (Formula interpretation) Single sequents are interpreted as formulas of the logic as follows:

$$i(A_1, ..., A_n, \langle\Sigma_1\rangle_{a_1}^{\mathbb{E}}, ..., \langle\Sigma_m\rangle_{a_m}^{\mathbb{E}}, \langle\Pi_1\rangle_{b_1}^{\mathbb{C}}, ..., \langle\Pi_k\rangle_{b_k}^{\mathbb{C}} \Rightarrow B_1, ..., B_\ell)$$
$$=$$
$$\bigwedge_{i\leq n} A_i \wedge \bigwedge_{j\leq m} \mathbb{E}_{a_j} \bigwedge \Sigma_j \wedge \bigwedge_{s\leq k} \mathbb{C}_{a_s} \bigwedge \Pi_s \rightarrow \bigvee_{t\leq\ell} B_t.$$

Definition 3.3 (Semantic interpretation) We say that a sequent S is *valid* in a bi-neighbourhood model $\mathcal{M}$, denoted $\mathcal{M} \models S$, if for all $w \in \mathcal{M}$, $\mathcal{M}, w \Vdash i(S)$. We say that a hypersequent H is *valid* in $\mathcal{M}$, denoted $\mathcal{M} \models H$, if $\mathcal{M} \models S$ for some $S \in H$.

The rules of the hypersequent calculus $\mathbf{HS_{ELG}}$ are presented in Fig. 2. They are expressed in the cumulative version: the principal formulas or blocks are copied into the premiss(es). This allows us to extract a countermodel from a single saturated hypersequent. The propositional rules are just the hypersequent versions of the ordinary corresponding sequent rules (we omit the rules for $\neg$, $\vee$, $\rightarrow$, which are standard). As usual, initial sequents init are restricted to propositional variables, but it is easy to see that $G \mid A, \Gamma \Rightarrow \Delta, A$ is derivable for every A. Similarly to propositional connectives, $\mathbb{E}$- and $\mathbb{C}$-formulas are handled by separate left and right rules. The rules $\mathsf{R}_{\mathbb{E}}$ and $\mathsf{R}_{\mathbb{C}}$ have multiple premisses, but their number is fixed by the cardinality of the principal blocks $\langle\Sigma\rangle_i^{\mathbb{E}}$ and $\langle\Sigma\rangle_i^{\mathbb{C}}$. For every axiom of **ELG** there is a corresponding rule in the calculus. Blocks have a central role in all modal rules. Observe in particular that $\mathbb{E}$-blocks can be merged by means of the rule $\mathsf{C}_{\mathbb{E}}$, but there is no analogous rule for $\mathbb{C}$-blocks. However, once complex $\mathbb{E}$-blocks are created, they can be converted into $\mathbb{C}$-blocks by means of the rule $\mathsf{Int}_{\mathbb{EC}}$. In Fig. 3 we show two examples of derivation in $\mathbf{HS_{ELG}}$.

Proposition 3.4 (Soundness) *If H is derivable in $\mathbf{HS_{ELG}}$, then it is valid in all bi-neighbourhood models for* **ELG**.

$$\mathsf{init}\ \frac{}{G \mid \Gamma, p \Rightarrow p, \Delta} \qquad \mathsf{L}\bot\ \frac{}{G \mid \Gamma, \bot \Rightarrow \Delta} \qquad \mathsf{R}\top\ \frac{}{G \mid \Gamma \Rightarrow \top, \Delta}$$

$$\mathsf{L}\wedge\ \frac{G \mid \Gamma, A \wedge B, A, B \Rightarrow \Delta}{G \mid \Gamma, A \wedge B \Rightarrow \Delta} \qquad \mathsf{R}\wedge\ \frac{G \mid \Gamma \Rightarrow A, A \wedge B, \Delta \qquad G \mid \Gamma \Rightarrow B, A \wedge B, \Delta}{G \mid \Gamma \Rightarrow A \wedge B, \Delta}$$

$$\mathsf{L}_{\mathbb{E}}\ \frac{G \mid \Gamma, \mathbb{E}_i A, \langle A \rangle_i^{\mathbb{E}} \Rightarrow \Delta}{G \mid \Gamma, \mathbb{E}_i A \Rightarrow \Delta} \qquad \mathsf{L}_{\mathbb{C}}\ \frac{G \mid \Gamma, \mathbb{C}_i A, \langle A \rangle_i^{\mathbb{C}} \Rightarrow \Delta}{G \mid \Gamma, \mathbb{C}_i A \Rightarrow \Delta}$$

$$\mathsf{Int}_{\mathbb{E}\mathbb{C}}\ \frac{G \mid \Gamma, \langle \Sigma \rangle_i^{\mathbb{E}}, \langle \Sigma \rangle_i^{\mathbb{C}} \Rightarrow \Delta}{G \mid \Gamma, \langle \Sigma \rangle_i^{\mathbb{E}} \Rightarrow \Delta}$$

$$\mathsf{R}_{\mathbb{E}}\ \frac{G \mid \Gamma, \langle \Sigma \rangle_i^{\mathbb{E}} \Rightarrow \mathbb{E}_i A, \Delta \mid \Sigma \Rightarrow A \qquad \{G \mid \Gamma, \langle \Sigma \rangle_i^{\mathbb{E}} \Rightarrow \mathbb{E}_i A, \Delta \mid A \Rightarrow B\}_{B \in \Sigma}}{G \mid \Gamma, \langle \Sigma \rangle_i^{\mathbb{E}} \Rightarrow \mathbb{E}_i A, \Delta}$$

$$\mathsf{R}_{\mathbb{C}}\ \frac{G \mid \Gamma, \langle \Sigma \rangle_i^{\mathbb{C}} \Rightarrow \mathbb{C}_i A, \Delta \mid \Sigma \Rightarrow A \qquad \{G \mid \Gamma, \langle \Sigma \rangle_i^{\mathbb{C}} \Rightarrow \mathbb{C}_i A, \Delta \mid A \Rightarrow B\}_{B \in \Sigma}}{G \mid \Gamma, \langle \Sigma \rangle_i^{\mathbb{C}} \Rightarrow \mathbb{C}_i A, \Delta}$$

$$\mathsf{C}_{\mathbb{E}}\ \frac{G \mid \Gamma, \langle \Sigma \rangle_i^{\mathbb{E}}, \langle \Pi \rangle_i^{\mathbb{E}}, \langle \Sigma, \Pi \rangle_i^{\mathbb{E}} \Rightarrow \Delta}{G \mid \Gamma, \langle \Sigma \rangle_i^{\mathbb{E}}, \langle \Pi \rangle_i^{\mathbb{E}} \Rightarrow \Delta} \qquad \mathsf{T}_{\mathbb{E}}\ \frac{G \mid \Gamma, \langle \Sigma \rangle_i^{\mathbb{E}}, \Sigma \Rightarrow \Delta}{G \mid \Gamma, \langle \Sigma \rangle_i^{\mathbb{E}} \Rightarrow \Delta}$$

$$\mathsf{Q}_{\mathbb{C}}\ \frac{\{G \mid \Gamma, \langle \Sigma \rangle_i^{\mathbb{C}} \Rightarrow \Delta \mid \ \Rightarrow B\}_{B \in \Sigma}}{G \mid \Gamma, \langle \Sigma \rangle_i^{\mathbb{C}} \Rightarrow \Delta} \qquad \mathsf{P}_{\mathbb{C}}\ \frac{G \mid \Gamma, \langle \Sigma \rangle_i^{\mathbb{C}} \Rightarrow \Delta \mid \Sigma \Rightarrow}{G \mid \Gamma, \langle \Sigma \rangle_i^{\mathbb{C}} \Rightarrow \Delta}$$

Fig. 2. The calculus $\mathbf{HS_{ELG}}$.

$$\cfrac{\cfrac{\cfrac{\mathbb{E}_i A, \langle A \rangle_i^{\mathbb{E}}, \langle A \rangle_i^{\mathbb{C}} \Rightarrow \mathbb{C}_i A \mid A \Rightarrow A \qquad \mathbb{E}_i A, \langle A \rangle_i^{\mathbb{E}}, \langle A \rangle_i^{\mathbb{C}} \Rightarrow \mathbb{C}_i A \mid A \Rightarrow A}{\mathbb{E}_i A, \langle A \rangle_i^{\mathbb{E}}, \langle A \rangle_i^{\mathbb{C}} \Rightarrow \mathbb{C}_i A}\,\mathsf{R}_{\mathbb{C}}}{\mathbb{E}_i A, \langle A \rangle_i^{\mathbb{E}} \Rightarrow \mathbb{C}_i A}\,\mathsf{Int}_{\mathbb{E}\mathbb{C}}}{\mathbb{E}_i A \Rightarrow \mathbb{C}_i A}\,\mathsf{L}_{\mathbb{E}}$$

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\ldots, \langle A, B \rangle_i^{\mathbb{E}} \Rightarrow \mathbb{E}_i(A \wedge B) \mid A, B \Rightarrow A \wedge B \qquad \ldots \mid A \wedge B \Rightarrow A \qquad \ldots \mid A \wedge B \Rightarrow B}{\mathbb{E}_i A \wedge \mathbb{E}_i B, \mathbb{E}_i A, \mathbb{E}_i B, \langle A \rangle_i^{\mathbb{E}}, \langle B \rangle_i^{\mathbb{E}}, \langle A, B \rangle_i^{\mathbb{E}} \Rightarrow \mathbb{E}_i(A \wedge B)}\,\mathsf{R}_{\mathbb{E}}}{\mathbb{E}_i A \wedge \mathbb{E}_i B, \mathbb{E}_i A, \mathbb{E}_i B, \langle A \rangle_i^{\mathbb{E}}, \langle B \rangle_i^{\mathbb{E}} \Rightarrow \mathbb{E}_i(A \wedge B)}\,\mathsf{C}_{\mathbb{E}}}{\mathbb{E}_i A \wedge \mathbb{E}_i B, \mathbb{E}_i A, \mathbb{E}_i B, \langle A \rangle_i^{\mathbb{E}} \Rightarrow \mathbb{E}_i(A \wedge B)}\,\mathsf{L}_{\mathbb{E}}}{\mathbb{E}_i A \wedge \mathbb{E}_i B, \mathbb{E}_i A, \mathbb{E}_i B \Rightarrow \mathbb{E}_i(A \wedge B)}\,\mathsf{L}_{\mathbb{E}}}{\mathbb{E}_i A \wedge \mathbb{E}_i B \Rightarrow \mathbb{E}_i(A \wedge B)}\,\mathsf{L}\wedge$$

Fig. 3. Derivations of axioms $\mathrm{Int}_{\mathbb{E}\mathbb{C}}$ and $\mathrm{C}_{\mathbb{E}}$ in $\mathbf{HS_{ELG}}$.

Proof. As usual, we have to show that the initial sequents are valid, and that whenever the premiss(es) of a rule are valid, so is the conclusion. We show the following illustrative cases.

($\mathsf{R}_{\mathbb{E}}$) Assume $\mathcal{M} \models G \mid \Gamma, \langle \Sigma \rangle_i^{\mathbb{E}} \Rightarrow \mathbb{E}_i A, \Delta \mid \Sigma \Rightarrow A$ and $\mathcal{M} \models G \mid \Gamma, \langle \Sigma \rangle_i^{\mathbb{E}} \Rightarrow \mathbb{E}_i A, \Delta \mid A \Rightarrow B$ for all $B \in \Sigma$. Then (i) $\mathcal{M} \models G$, or (ii) $\mathcal{M} \models \Gamma, \langle \Sigma \rangle_i^{\mathbb{E}} \Rightarrow \mathbb{E}_i A, \Delta$, or (iii) $\mathcal{M} \models \Sigma \Rightarrow A$ and $\mathcal{M} \models A \Rightarrow B$ for all $B \in \Sigma$. If (i) or (ii) we are done. If (iii), then $\mathcal{M} \models \bigwedge \Sigma \rightarrow A$ and $\mathcal{M} \models A \rightarrow B$ for all $B \in \Sigma$, that is $\mathcal{M} \models \bigwedge \Sigma \leftrightarrow A$. Since $\mathrm{RE}_{\mathbb{E}}$ is valid, $\mathcal{M} \models \mathbb{E}_i \bigwedge \Sigma \to \mathbb{E}_i A = i(\langle \Sigma \rangle_i^{\mathbb{E}} \Rightarrow \mathbb{E}_i A)$. Thus $\mathcal{M} \models \Gamma, \langle \Sigma \rangle_i^{\mathbb{E}} \Rightarrow \mathbb{E}_i A, \Delta$.

($\mathsf{Int}_{\mathbb{EC}}$) Assume $\mathcal{M} \models G \mid \Gamma, \langle \Sigma \rangle_i^{\mathbb{E}}, \langle \Sigma \rangle_i^{\mathbb{C}} \Rightarrow \Delta$. Then $\mathcal{M} \models G$ or $\mathcal{M} \models \Gamma, \langle \Sigma \rangle_i^{\mathbb{E}}, \langle \Sigma \rangle_i^{\mathbb{C}} \Rightarrow \Delta$. In the first case we are done. In the second case, $\mathcal{M} \models i(\Gamma, \langle \Sigma \rangle_i^{\mathbb{E}}, \langle \Sigma \rangle_i^{\mathbb{C}} \Rightarrow \Delta)$, which is equivalent to $\mathbb{E}_i \bigwedge \Sigma \wedge \mathbb{C}_i \bigwedge \Sigma \to i(\Gamma \Rightarrow \Delta)$. By the validity of axiom $\mathrm{Int}_{\mathbb{EC}}$, this is in turn equivalent to $\mathbb{E}_i \bigwedge \Sigma \to i(\Gamma \Rightarrow \Delta)$. Therefore $\mathcal{M} \models i(\Gamma, \langle \Sigma \rangle_i^{\mathbb{E}} \Rightarrow \Delta)$. □

We now investigate the structural properties of our calculus, and show that it is complete with respect to the axiomatisation. A purely syntactic completeness proof is significant because it is independent from the choice of any specific semantics. As usual, this proof requires to show the admissibility of the cut rule, that we formulate as follows:

$$\mathsf{cut}\ \frac{G \mid \Gamma \Rightarrow \Delta, A \qquad G \mid A, \Gamma \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \Delta}$$

This means that whenever the premisses of cut are derivable, the conclusion is also derivable. In turn, admissibility of cut depends upon the admissibility of the structural rules of weakening and contraction, that in the hypersequent framework must be formulated both in their internal and in their external variants as follows:

Proposition 3.5 (Admissibility of structural rules) *The following rules are admissible in* $\mathbf{HS_{ELG}}$*, where* ϕ *is any formula* A *or block* $\langle \Sigma \rangle_i^{\mathbb{E}}$ *or* $\langle \Sigma \rangle_i^{\mathbb{C}}$*:*

$$\mathsf{Lwk}\ \frac{G \mid \Gamma \Rightarrow \Delta}{G \mid \phi, \Gamma \Rightarrow \Delta} \qquad \mathsf{Lctr}\ \frac{G \mid \phi, \phi, \Gamma \Rightarrow \Delta}{G \mid \phi, \Gamma \Rightarrow \Delta} \qquad \mathsf{Bctr}\ \frac{G \mid \langle A, A, \Sigma \rangle, \Gamma \Rightarrow \Delta}{G \mid \langle A, \Sigma \rangle, \Gamma \Rightarrow \Delta}$$

$$\mathsf{Rwk}\ \frac{G \mid \Gamma \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \Delta, A} \qquad \mathsf{Rctr}\ \frac{G \mid \Gamma \Rightarrow \Delta, A, A}{G \mid \Gamma \Rightarrow \Delta, A}$$

$$\mathsf{Ewk}\ \frac{G}{G \mid \Gamma \Rightarrow \Delta} \qquad \mathsf{Ectr}\ \frac{G \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \Delta}$$

The proof of admissibility of weakening and contraction is standard by induction on the derivation of the premisses. Observe that as an immediate consequence of the admissibility of weakening all rules are invertible, which means that whenever the conclusion of a rule is derivable, so are the premisses. This is important because if a formula is derivable we get a derivation no matter the order in which the rules are applied (see Sec. 4).

By contrast, the proof of admissibility of cut is a bit more intricate and deserves more attention. We shall prove simultaneously the admissibility of cut and of the following rule sub, which states that a formula A inside one or more blocks can be replaced by any equivalent set of formulas Σ:

$$\mathsf{sub}\ \frac{G \mid \Sigma \Rightarrow A \qquad \{G \mid A \Rightarrow B\}_{B \in \Sigma} \qquad G \mid \overrightarrow{\langle A^n, \Pi \rangle_i^{\mathbb{E}}}, \overrightarrow{\langle A^m, \Omega \rangle_j^{\mathbb{C}}}, \Gamma \Rightarrow \Delta}{G \mid \overrightarrow{\langle \Sigma^n, \Pi \rangle_i^{\mathbb{E}}}, \overrightarrow{\langle \Sigma^m, \Omega \rangle_j^{\mathbb{C}}}, \Gamma \Rightarrow \Delta}$$

where for instance $\overrightarrow{\langle A^n, \Pi \rangle_i^{\mathbb{E}}}$ stays for $\langle A^{n_1}, \Pi_1 \rangle_{i_1}^{\mathbb{E}}, ..., \langle A^{n_k}, \Pi_k \rangle_{i_k}^{\mathbb{E}}$, and A^{n_ℓ} is a compact way to denote n_ℓ occurrences of A. In the proof we use the following definition of weight of formulas and blocks.

Definition 3.6 (Weight of formulas and blocks) The weight of formulas and blocks is recursively defined as follows: $\mathsf{w}(\bot) = \mathsf{w}(\top) = \mathsf{w}(p) = 0$; $\mathsf{w}(A \wedge B) = \mathsf{w}(A \vee B) = \mathsf{w}(A \to B) = \mathsf{w}(A) + \mathsf{w}(B) + 1$; $\mathsf{w}(\langle A_1, ..., A_k\rangle_i^{\mathbb{E}}) = \mathsf{w}(\langle A_1, ..., A_k\rangle_j^{\mathbb{C}}) = max_{1\leq n \leq k}\{\mathsf{w}(A_n)\} + 1$, $\mathsf{w}(\mathbb{E}_i A) = \mathsf{w}(\mathbb{C}_i A) = \mathsf{w}(A) + 2$.

Theorem 3.7 (Cut elimination) *The rules* cut *and* sub *are admissible in* $\mathbf{HS_{ELG}}$.

Sketch of Proof. Let $Cut(c, h)$ mean that all applications of cut of height h on a cut formula of weight c are admissible, and $Sub(c)$ mean that all applications of sub where A has weight c are admissible. Then the theorem is a consequence of the following claims: **(A)** $\forall c.Cut(c, 0)$; **(B)** $\forall h.Cut(0, h)$; **(C)** $\forall c.(\forall h.Cut(c, h) \to Sub(c))$; **(D)** $\forall c.\forall h.\ ((\forall c' < c.(Sub(c') \wedge \forall h'.Cut(c', h')) \wedge \forall h'' < h.Cut(c, h'')) \to Cut(c, h))$. The proof is in the Appendix. □

As a consequence of admissibility of cut we can prove the following completeness theorem.

Theorem 3.8 (Axiomatic completeness) *If A is derivable in* **ELG***, then* $\Rightarrow A$ *is derivable in* $\mathbf{HS_{ELG}}$.

Proof. All modal axioms and rules of **ELG** are derivable in $\mathbf{HS_{ELG}}$. As examples, in Fig. 3 we have shown the derivations of axioms $\mathrm{Int}_{\mathbb{EC}}$ and $\mathrm{C}_{\mathbb{E}}$. Moreover, the rule $\mathrm{RE}_{\mathbb{E}}$ (and analogously the rule $\mathrm{RE}_{\mathbb{C}}$) is derived as follows:

$$\dfrac{\dfrac{\dfrac{A \Rightarrow B}{\mathbb{E}_i A, \langle A\rangle_i^{\mathbb{E}} \Rightarrow \mathbb{E}_i B \mid A \Rightarrow B}\,\mathsf{Ewk} \qquad \dfrac{B \Rightarrow A}{\mathbb{E}_i A, \langle A\rangle_i^{\mathbb{E}} \Rightarrow \mathbb{E}_i B \mid B \Rightarrow A}\,\mathsf{Ewk}}{\mathbb{E}_i A, \langle A\rangle_i^{\mathbb{E}} \Rightarrow \mathbb{E}_i B}\,\mathsf{R}_{\mathbb{E}}}{\mathbb{E}_i A \Rightarrow \mathbb{E}_i B}\,\mathsf{L}_{\mathbb{E}}$$

The derivation contains applications of Ewk, which has been proved admissible. Finally, Modus Ponens is simulated by cut, which has been proved admissible, in the usual way. □

As mentioned, hypersequents are not strictly necessary for making derivations, and in particular one can show that a hypersequent is derivable in $\mathbf{HS_{ELG}}$ if and only if one of its components is derivable. However, the use of hypersequents allows us to obtain a calculus where all rules are invertible, which entails that the order of rule applications does not matter: essentially, modulo the order of rule applications, every formula has a *single* derivation, or a *single* failed proof, whence in particular proof search does not require backtracking. Moreover, hypersequents are crucial for a direct computation of countermodels from every single unprovable hypersequent occurring as a leaf of a failed derivation. We shall see all this in the next section.

4 Proof search and countermodel extraction

In this section, we define a procedure for checking the validity/derivability of formulas in Elgesem's logic by means of our hypersequent calculus. The procedure is based on a simple root-first proof search strategy. We show that the strategy always terminates and constructs a derivation for every valid formula. Moreover, we show that whenever the proof fails it possible to directly extract a countermodel of the non-valid formula. The strategy is based on the notion of saturation. Intuitively, a saturated hypersequent is such that the backward application of any rule to it cannot add any information, in the sense that one of the premisses of the rule is already included in the hypersequent.

Definition 4.1 (Saturated hypersequent) Let $H = \Gamma_1 \Rightarrow \Delta_1 \mid ... \mid \Gamma_k \Rightarrow \Delta_k$ be a hypersequent occurring in a proof for H'. The saturation conditions associated to each application of a rule of $\mathbf{HS_{ELG}}$ are as follows:

- Unprovability: (init) $\Gamma_n \cap \Delta_n = \emptyset$. $(\bot_\mathsf{L})$ $\bot \notin \Gamma_n$. $(\top_\mathsf{R})$ $\top \notin \Delta_n$.
- Propositional rules: $(\wedge_\mathsf{L})$ If $A \wedge B \in \Gamma_n$, then $A \in \Gamma_n$ and $B \in \Gamma_n$. $(\wedge_\mathsf{R})$ If $A \wedge B \in \Delta_n$, then $A \in \Delta_n$ or $B \in \Delta_n$. Analogous for the rules for $\neg$, $\vee$, $\rightarrow$.
- Modal rules: $(\mathsf{L}_\mathbb{E})$ If $\mathbb{E}_i A \in \Gamma_n$, then $\langle A \rangle_i^\mathbb{E} \in \Gamma_n$. $(\mathsf{R}_\mathbb{E})$ If $\Gamma, \langle \Sigma \rangle_i^\mathbb{E} \Rightarrow \mathbb{E}_i B, \Delta$ is in H, then there is $\Gamma', \Sigma \Rightarrow B, \Delta'$ in H or there is $\Gamma', B \Rightarrow A, \Delta'$ in H for some $A \in \Sigma$. $(\mathsf{L}_\mathbb{C})$ and $(\mathsf{R}_\mathbb{C})$ are analogous. $(\mathsf{C}_\mathbb{E})$ If $\langle \Sigma \rangle_i^\mathbb{E}, \langle \Pi \rangle_i^\mathbb{E} \in \Gamma_n$, then there is $\langle \Omega \rangle_i^\mathbb{E} \in \Gamma_n$ such that $\mathsf{set}(\Sigma, \Pi) = \mathsf{set}(\Omega)$. $(\mathsf{T}_\mathbb{E})$ If $\langle \Sigma \rangle_i^\mathbb{E} \in \Gamma_n$, then $\mathsf{set}(\Sigma) \subseteq \Gamma_n$. $(\mathsf{Q}_\mathbb{C})$ If $\Gamma, \langle \Sigma \rangle_i^\mathbb{C} \Rightarrow \Delta$ is in H, then there is $\Gamma' \Rightarrow B, \Delta'$ in H for some $B \in \Sigma$. $(\mathsf{P}_\mathbb{C})$ If $\Gamma, \langle \Sigma \rangle_i^\mathbb{C} \Rightarrow \Delta$ is in H, then there is $\Gamma' \Rightarrow \Delta'$ in H such that $\mathsf{set}(\Sigma) \subseteq \Gamma'$. $(\mathsf{Int}_{\mathbb{EC}})$ If $\langle \Sigma \rangle_i^\mathbb{E} \in \Gamma_n$, then there is $\langle \Omega \rangle_i^\mathbb{C} \in \Gamma_n$ such that $\mathsf{set}(\Sigma) = \mathsf{set}(\Omega)$.

We say that H is saturated with respect to an application of a rule R if it satisfies the corresponding saturation condition (R) for that particular rule application, and that it is saturated with respect to $\mathbf{HS_{ELG}}$ if it is saturated with respect to every possible application of any rule of $\mathbf{HS_{ELG}}$.

The *proof search strategy* is simple: (i) do not apply any rule to initial sequents, and (ii) do not apply a rule to a hypersequent which is already saturated with respect to that particular application of that rule.

The strategy essentially amounts to avoiding applications of rules that do not add any additional information to the hypersequents. We can prove that this strategy leads to a terminating proof search algorithm.

Proposition 4.2 (Termination of proof search) *Every branch of a proof of a hypersequent H built in accordance with the strategy is finite. Thus, the proof search procedure for H always terminates. Moreover, every branch ends either with an initial hypersequent or a saturated one.*

Proof. Let $\mathfrak{P}$ be a proof of H. Then all formulas occurring in $\mathfrak{P}$ (both inside and outside blocks) are subformulas of formulas of H, so they are finitely many. Moreover, saturation conditions prevent duplications of the same formulas (both inside and outside blocks) and same blocks. Therefore every branch of $\mathfrak{P}$ can contain only finitely many hypersequents. $\square$

Hypersequents occurring in a proof of H can be exponentially large with respect to the size of H. This is due to the presence of the rule $\mathsf{C}_\mathbb{E}$ that, given n formulas $\mathbb{E}_i A_1, \ldots, \mathbb{E}_i A_n$, allows one to build a block for every subset of $\{A_1, ..., A_n\}$. In this respect, our decision procedure does not match the PSPACE complexity upper bound established for Elgesem's logic by Schröder and Pattinson [10] and Troquard [11].

An optimal calculus could be obtained either by considering the sequent calculus in [9], or (similarly to [3]) by reformulating the rules in Fig. 2 in such a way that the principal formulas are *not copied* into the premisses. However, in this way we would lose the invertibility of the rules, whence the possibility to directly extract countermodels from single failed proofs. The situation is analogous to the one of modal logic **K**: while a PSPACE complexity upper bound can be obtain with the sequent calculus, the same is not possible with a calculus with only invertible rules allowing for direct countermodel extraction of non-valid formulas. This essentially shows the existence of a necessary trade-off in the logic **ELG** between the optimal complexity of the calculus and the possibility to directly extract countermodels from failed proofs.

We now show how to directly build a countermodel in the bi-neighbourhood semantics from a saturated hypersequent.

Definition 4.3 (Countermodel construction) Let H be a saturated hypersequent occurring in a proof for H'. Moreover, let $e : \mathbb{N} \longrightarrow H$ be an enumeration of the components of H. Given e, we can write H as $\Gamma_1 \Rightarrow \Delta_1 \mid ... \mid \Gamma_k \Rightarrow \Delta_k$. The model $\mathcal{M} = \langle \mathcal{W}, \mathcal{N}, \mathcal{V} \rangle$ is defined as follows:

- $\mathcal{W} = \{n \mid \Gamma_n \Rightarrow \Delta_n \in H\}$.
- $\mathcal{V}(p) = \{n \mid p \in \Gamma_n\}$.
- For every block $\langle\Sigma\rangle_i^{\mathbb{E}}$ or $\langle\Sigma\rangle_i^{\mathbb{C}}$ occurring in a component $\Gamma_m \Rightarrow \Delta_m$ of H,
$$\Sigma^+ = \{n \in \mathcal{W} \mid \mathsf{set}(\Sigma) \subseteq \Gamma_n\} \text{ and } \Sigma^- = \{n \in \mathcal{W} \mid \Sigma \cap \Delta_n \neq \emptyset\}.$$
- For every $i \in \mathcal{A}$ and every $n \in \mathcal{W}$,
$$\mathcal{N}_i^{\mathbb{E}}(n) = \{(\Sigma^+, \Sigma^-) \mid \langle\Sigma\rangle_i^{\mathbb{E}} \in \Gamma_n\} \text{ and } \mathcal{N}_i^{\mathbb{C}}(n) = \{(\Sigma^+, \Sigma^-) \mid \langle\Sigma\rangle_i^{\mathbb{C}} \in \Gamma_n\}.$$

Lemma 4.4 *Let $\mathcal{M}$ be defined as in Def. 4.3. Then for every A, $\langle\Sigma\rangle_i^{\mathbb{E}}$, $\langle\Pi\rangle_j^{\mathbb{C}}$ and every $n \in \mathcal{W}$, we have: If $A \in \Gamma_n$, then $n \Vdash A$; if $\langle\Sigma\rangle_i^{\mathbb{E}} \in \Gamma_n$, then $n \Vdash \mathbb{E}_i \bigwedge \Sigma$; if $\langle\Pi\rangle_j^{\mathbb{C}} \in \Gamma_n$, then $n \Vdash \mathbb{C}_j \bigwedge \Pi$; and if $A \in \Delta_n$, then $n \nVdash A$. Moreover, $\mathcal{M}$ is a bi-neighbourhood model for* **ELG**.

Proof. The first claim is proved by mutual induction on A and $\langle\Sigma\rangle_i^{\mathbb{E}}, \langle\Sigma\rangle_i^{\mathbb{C}}$. We only consider the inductive cases of modal formulas and blocks.

($\langle\Sigma\rangle_i^{\mathbb{E}} \in \Gamma_n$) By definition, $(\Sigma^+, \Sigma^-) \in \mathcal{N}_i^{\mathbb{E}}(n)$. We show that $\Sigma^+ \subseteq [\![\bigwedge \Sigma]\!]$ and $\Sigma^- \subseteq [\![\neg \bigwedge \Sigma]\!]$, which implies $n \Vdash \mathbb{E}_i \bigwedge \Sigma$. If $m \in \Sigma^+$, then $\mathsf{set}(\Sigma) \subseteq \Gamma_m$. By i.h. $m \Vdash A$ for all $A \in \Sigma$, then $m \Vdash \bigwedge \Sigma$. If $m \in \Sigma^-$, then there is $B \in \Sigma \cap \Delta_m$. By i.h. $m \nVdash B$, then $m \nVdash \bigwedge \Sigma$.

($\mathbb{E}_i B \in \Gamma_n$) By saturation of rule $\mathsf{L}_{\mathbb{E}}$, $\langle B\rangle_i^{\mathbb{E}} \in \Gamma_n$. Then by i.h. $n \Vdash \mathbb{E}_i B$.

($\mathbb{E}_i B \in \Delta_n$) Assume $(\alpha, \beta) \in \mathcal{N}_i^{\mathbb{E}}(n)$. Then there is $\langle\Sigma\rangle_i^{\mathbb{E}} \in \Gamma_n$ such that $\Sigma^+ = \alpha$ and $\Sigma^- = \beta$. By saturation of rule $\mathsf{R}_{\mathbb{E}}$, there is $m \in \mathcal{W}$ such that $\Sigma \subseteq \Gamma_m$ and $B \in \Delta_m$, or there is $m \in \mathcal{W}$ such that $\Sigma \cap \Delta_m \neq \emptyset$ and $B \in \Gamma_m$. In the first case, $m \in \Sigma^+ = \alpha$ and by i.h. $m \nVdash B$, thus $\alpha \not\subseteq [\![B]\!]$. In the second case, $m \in \Sigma^- = \beta$ and by i.h. $m \Vdash B$, thus $\beta \not\subseteq [\![\neg B]\!]$. Therefore $n \not\Vdash \mathbb{E}_i B$.

For blocks $\langle\Sigma\rangle_i^{\mathbb{C}}$ and formulas $\mathbb{C}_i B$ the proof is analogous. Now we prove that $\mathcal{M}$ is a model for **ELG**.

($\mathrm{C}_{\mathbb{E}}$) Assume that $(\alpha, \beta), (\gamma, \delta) \in \mathcal{N}_i^{\mathbb{E}}(n)$. Then there are $\langle\Sigma\rangle_i^{\mathbb{E}}, \langle\Pi\rangle_i^{\mathbb{E}} \in \Gamma_n$ such that $\Sigma^+ = \alpha$, $\Sigma^- = \beta$, $\Pi^+ = \gamma$ and $\Pi^- = \delta$. By saturation of rule $\mathsf{C}_{\mathbb{E}}$, there is $\langle\Omega\rangle \in \Gamma_n$ such that $\mathsf{set}(\Omega) = \mathsf{set}(\Sigma, \Pi)$, thus $(\Omega^+, \Omega^-) \in \mathcal{N}_i^{\mathbb{E}}(n)$. We show that (i) $\Omega^+ = \alpha \cap \gamma$ and (ii) $\Omega^- = \beta \cup \delta$. (i) $m \in \Omega^+$ iff $\mathsf{set}(\Omega) = \mathsf{set}(\Sigma, \Pi) \subseteq \Gamma_m$ iff $\mathsf{set}(\Sigma) \subseteq \Gamma_m$ and $\mathsf{set}(\Pi) \subseteq \Gamma_m$ iff $m \in \Sigma^+ = \alpha$ and $m \in \Pi^+ = \gamma$ iff $m \in \alpha \cap \gamma$. (ii) $m \in \Omega^-$ iff $\Omega \cap \Delta_m \neq \emptyset$ iff $\Sigma, \Pi \cap \Delta_m \neq \emptyset$ iff $\Sigma \cap \Delta_m \neq \emptyset$ or $\Pi \cap \Delta_m \neq \emptyset$ iff $m \in \Sigma^- = \beta$ or $m \in \Pi^- = \delta$ iff $m \in \beta \cup \delta$.

($\mathrm{Int}_{\mathbb{E}\mathbb{C}}$, $\mathrm{T}_{\mathbb{E}}$) If $(\alpha, \beta) \in \mathcal{N}_i^{\mathbb{E}}(n)$, then there is $\langle\Sigma\rangle_i^{\mathbb{E}} \in \Gamma_n$ such that $\Sigma^+ = \alpha$ and $\Sigma^- = \beta$. By saturation of rule $\mathsf{T}_{\mathbb{E}}$, $\mathsf{set}(\Sigma) \subseteq \Gamma_n$, then $n \in \Sigma^+ = \alpha$. Moreover, by saturation of rule $\mathsf{Int}_{\mathbb{E}\mathbb{C}}$, then there is $\langle\Omega\rangle_i^{\mathbb{C}} \in \Gamma_n$ such that $\mathsf{set}(\Sigma) = \mathsf{set}(\Omega)$. Then $(\Omega^+, \Omega^-) = (\Sigma^+, \Sigma^-) = (\alpha, \beta) \in \mathcal{N}_i^{\mathbb{C}}(n)$.

($\mathrm{P}_{\mathbb{C}}$, $\mathrm{Q}_{\mathbb{C}}$) If $(\alpha, \beta) \in \mathcal{N}_i^{\mathbb{C}}(n)$, then there is $\langle\Sigma\rangle_i^{\mathbb{C}} \in \Gamma_n$ such that $\Sigma^+ = \alpha$ and $\Sigma^- = \beta$. By saturation of rule $\mathsf{P}_{\mathbb{C}}$, there is $m \in \mathcal{W}$ such that $\mathsf{set}(\Sigma) \subseteq \Gamma_m$. Then $m \in \Sigma^+ = \alpha$, that is $\alpha \neq \emptyset$. Moreover, by saturation of rule $\mathsf{Q}_{\mathbb{C}}$, there is $\ell \in \mathcal{W}$ such that $\Sigma \cap \Delta_\ell \neq \emptyset$. Then $\ell \in \Sigma^- = \beta$, that is $\beta \neq \emptyset$. □

Observe that since all rules are cumulative, the countermodel $\mathcal{M}$ of H is also a countermodel of the root hypersequent H'. Then for every hypersequent we either get a derivation (if the hypersequent is valid) or obtain a countermodel. This entails the following theorem.

Theorem 4.5 (Semantic completeness) *If H is valid in all bi-neighbourhood models for* **ELG**, *then it is derivable in* $\mathbf{HS_{ELG}}$.

The proof search procedure for the calculus $\mathbf{HS_{ELG}}$ can be used to automatically and constructively check the validity/derivability of formulas in Elgesem logic. For every formula, the proof search procedure either provides a derivation if the formula is valid, or returns a countermodel if it is not.

Example 4.6 (Failure of delegation) The treatment of delegation represents a main difference between Elgesem's account of agency and other accounts, such as for instance the one formalised by STIT logics. It is explicitly rejected by Elgesem [5]: "a person is normally not considered the agent of some consequence of his action if another agent interferes in the causal chain." For instance, we can say that having the car repaired is not the same as repairing the car by yourself. Let us represent Anna by a, Beatrice by b, and "repairing the car" by p. Then $\mathbb{E}_a\mathbb{E}_b p \to \mathbb{E}_a p$ expresses the sentence "If Anna gets Beatrice to repair her car, then Anna repairs her car". By using our calculus we can automatically obtain a countermodel of $\mathbb{E}_a\mathbb{E}_b p \to \mathbb{E}_a p$. First, in Fig. 4 we find a failed proof of $\mathbb{E}_a\mathbb{E}_b p \to \mathbb{E}_a p$ in $\mathbf{HS_{ELG}}$. Then we consider the following enumeration of the components of the saturated hypersequent: $1 \mapsto \langle\mathbb{E}_b p\rangle_a^{\mathbb{E}}, \langle p\rangle_b^{\mathbb{E}}, \langle\mathbb{E}_b p\rangle_a^{\mathbb{C}}, \langle p\rangle_b^{\mathbb{C}}, p, \mathbb{E}_b p, \mathbb{E}_a\mathbb{E}_b p \Rightarrow \mathbb{E}_a p$; $2 \mapsto p \Rightarrow \mathbb{E}_b p$; and $3 \mapsto\ \Rightarrow p$. We obtain the following countermodels:

<u>Bi-neighbourhood countermodel</u>: By applying the construction in Def. 4.3 we

$$
\dfrac{
\dfrac{\dfrac{\dfrac{}{\dots \mid \langle p\rangle^{\mathbb{E}}_b, p, \mathbb{E}_b p \Rightarrow p}\,\mathsf{init}}{\dots \mid \langle p\rangle^{\mathbb{E}}_b, \mathbb{E}_b p \Rightarrow p}\,\mathsf{T}_{\mathbb{E}}}{\dots \mid \mathbb{E}_b p \Rightarrow p}\,\mathsf{L}_{\mathbb{E}}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\textbf{saturated}\\ \langle \mathbb{E}_b p\rangle^{\mathbb{E}}_a, \langle p\rangle^{\mathbb{E}}_b, \langle \mathbb{E}_b p\rangle^{\mathbb{C}}_a, \langle p\rangle^{\mathbb{C}}_b, p, \mathbb{E}_b p, \mathbb{E}_a\mathbb{E}_b p \Rightarrow \mathbb{E}_a p \mid p \Rightarrow \mathbb{E}_b p \mid \Rightarrow p\end{array}}{\langle \mathbb{E}_b p\rangle^{\mathbb{E}}_a, \langle p\rangle^{\mathbb{E}}_b, \langle \mathbb{E}_b p\rangle^{\mathbb{C}}_a, \langle p\rangle^{\mathbb{C}}_b, p, \mathbb{E}_b p, \mathbb{E}_a\mathbb{E}_b p \Rightarrow \mathbb{E}_a p \mid p \Rightarrow \mathbb{E}_b p}\,\mathsf{Q}_{\mathbb{C}}}{\langle \mathbb{E}_b p\rangle^{\mathbb{E}}_a, \langle p\rangle^{\mathbb{E}}_b, \langle \mathbb{E}_b p\rangle^{\mathbb{C}}_a, p, \mathbb{E}_b p, \mathbb{E}_a\mathbb{E}_b p \Rightarrow \mathbb{E}_a p \mid p \Rightarrow \mathbb{E}_b p}\,\mathsf{Int}_{\mathbb{E}\mathbb{C}}}{\langle \mathbb{E}_b p\rangle^{\mathbb{E}}_a, \langle p\rangle^{\mathbb{E}}_b, p, \mathbb{E}_b p, \mathbb{E}_a\mathbb{E}_b p \Rightarrow \mathbb{E}_a p \mid p \Rightarrow \mathbb{E}_b p}\,\mathsf{Int}_{\mathbb{E}\mathbb{C}}}{\langle \mathbb{E}_b p\rangle^{\mathbb{E}}_a, \langle p\rangle^{\mathbb{E}}_b, \mathbb{E}_b p, \mathbb{E}_a\mathbb{E}_b p \Rightarrow \mathbb{E}_a p \mid p \Rightarrow \mathbb{E}_b p}\,\mathsf{T}_{\mathbb{E}}}{\langle \mathbb{E}_b p\rangle^{\mathbb{E}}_a, \mathbb{E}_b p, \mathbb{E}_a\mathbb{E}_b p \Rightarrow \mathbb{E}_a p \mid p \Rightarrow \mathbb{E}_b p}\,\mathsf{L}_{\mathbb{E}}}{\langle \mathbb{E}_b p\rangle^{\mathbb{E}}_a, \mathbb{E}_a\mathbb{E}_b p \Rightarrow \mathbb{E}_a p \mid p \Rightarrow \mathbb{E}_b p}\,\mathsf{T}_{\mathbb{E}}
}{
\dfrac{\langle \mathbb{E}_b p\rangle^{\mathbb{E}}_a, \mathbb{E}_a\mathbb{E}_b p \Rightarrow \mathbb{E}_a p}{\mathbb{E}_a\mathbb{E}_b p \Rightarrow \mathbb{E}_a p}\,\mathsf{L}_{\mathbb{E}}
}\,\mathsf{R}_{\mathbb{E}}
$$

Fig. 4. Failed proof in $\mathbf{HS_{ELG}}$.

$\mathrm{RE}_{\mathbb{E}}$	$\dfrac{A \leftrightarrow B}{\mathbb{E}_g A \leftrightarrow \mathbb{E}_g B}$	$\mathrm{RE}_{\mathbb{C}}$	$\dfrac{A \leftrightarrow B}{\mathbb{C}_g A \leftrightarrow \mathbb{C}_g B}$
$\mathrm{C}_{\mathbb{E}}$	$\mathbb{E}_g A \wedge \mathbb{E}_g B \to \mathbb{E}_g(A \wedge B)$	$\mathrm{Q}_{\mathbb{C}}$	$\neg\mathbb{C}_g\top$
$\mathrm{T}_{\mathbb{E}}$	$\mathbb{E}_g A \to A$	$\mathrm{P}_{\mathbb{C}}$	$\neg\mathbb{C}_g\bot$
$\mathrm{Int}^1_{\mathbb{E}\mathbb{C}}$	$\mathbb{E}_g A \to \mathbb{C}_g A$	$\mathrm{F}_{\mathbb{C}}$	$\neg\mathbb{C}_{\emptyset} A$
$\mathrm{Int}^2_{\mathbb{E}\mathbb{C}}$	$\mathbb{E}_{g_1} A \wedge \mathbb{E}_{g_2} B \to \mathbb{C}_{g_1 \cup g_2}(A \wedge B)$		

Fig. 5. Modal axioms and rules of Troquard's logic **COAL**.

obtain the bi-neighbourhood countermodel $\mathcal{M} = \langle \mathcal{W}, \mathcal{N}^{\mathbb{E}}_i, \mathcal{N}^{\mathbb{C}}_i, \mathcal{V}\rangle$, where $\mathcal{W} = \{1,2,3\}$; $\mathcal{V}(p) = \{1,2\}$; $\mathcal{N}^{\mathbb{E}}_a(1) = \mathcal{N}^{\mathbb{C}}_a(1) = \{(\{1\},\{2\})\}$ – since $\mathcal{N}^{\mathbb{E}}_a(1) = \mathcal{N}^{\mathbb{C}}_a(1) = \{(\mathbb{E}_b p^+, \mathbb{E}_b p^-)\}$, $\mathbb{E}_b p^+ = \{1\}$, and $\mathbb{E}_b p^- = \{2\}$ –; $\mathcal{N}^{\mathbb{E}}_b(1) = \mathcal{N}^{\mathbb{C}}_b(1) = \{(\{1,2\},\{3\})\}$ – since $\mathcal{N}^{\mathbb{E}}_b(1) = \mathcal{N}^{\mathbb{C}}_b(1) = \{(p^+, p^-)\}$, $p^+ = \{1,2\}$, and $p^- = \{3\}$ –; $\mathcal{N}^{\mathbb{E}}_i(n) = \mathcal{N}^{\mathbb{C}}_i(n) = \emptyset$ for $i = a, b$ and $n = 2, 3$.

<u>Neighbourhood countermodel</u>: By applying the transformation in Prop. 2.3 we obtain the neighbourhood countermodel $\mathcal{M} = \langle \mathcal{W}, \mathcal{N}^{\mathbb{E}}_i, \mathcal{N}^{\mathbb{C}}_i, \mathcal{V}\rangle$, where $\mathcal{W} = \{1,2,3\}$; $\mathcal{V}(p) = \{1,2\}$; $\mathcal{N}^{\mathbb{E}}_a(1) = \mathcal{N}^{\mathbb{C}}_a(1) = \{\{1\},\{1,3\}\}$; $\mathcal{N}^{\mathbb{E}}_b(1) = \mathcal{N}^{\mathbb{C}}_b(1) = \{\{1,2\}\}$; and $\mathcal{N}^{\mathbb{E}}_i(n) = \mathcal{N}^{\mathbb{C}}_i(n) = \emptyset$ for $i = a, b$ and $n = 2, 3$.

5 Extension to Troquard's coalition logic

A coalition version of Elgesem's logic is proposed by Troquard [11]. In Troquard's logic, called **COAL**, single agents are replaced by *groups* of agents. The aim is to represent what agents do and can do when acting in coalitions. The logic **COAL** is defined by extending classical propositional logic with the modal axioms and rules in Fig. 5.

Apart from $\mathrm{F}_{\mathbb{C}}$ and $\mathrm{Int}^2_{\mathbb{E}\mathbb{C}}$, the axioms and rules of **COAL** are just the coalition versions of the corresponding ones in **ELG**, with agents i replaced by groups g. The peculiar aspects of group agency are represented in **COAL** by the axioms $\mathrm{F}_{\mathbb{C}}$ and $\mathrm{Int}^2_{\mathbb{E}\mathbb{C}}$. In particular, the axiom $\mathrm{F}_{\mathbb{C}}$ expresses that the empty group cannot realise anything, whereas the axiom $\mathrm{Int}^2_{\mathbb{E}\mathbb{C}}$ says that if a group realises A and another group realises B, then by joining their forces they *could* realise both A and B. Observe that the axiom $\mathrm{Int}^1_{\mathbb{E}\mathbb{C}}$ is derivable from $\mathrm{Int}^2_{\mathbb{EC}}$. Nevertheless we keep it in the axiomatisation, as in [11], to keep the correspondence with the calculus where a specific rule for $\mathrm{Int}^1_{\mathbb{EC}}$ is needed to ensure the admissibility of contraction. As for **ELG**, we can define bi-neighbourhood models for **COAL**.

Definition 5.1 A bi-neighbourhood model for **COAL** is a tuple $\mathcal{M} = \langle \mathcal{W}, \mathcal{N}^{\mathbb{E}}_g, \mathcal{N}^{\mathbb{C}}_g, \mathcal{V} \rangle$, where in particular for every group of agents g, $\mathcal{N}^{\mathbb{E}}_g$ and $\mathcal{N}^{\mathbb{C}}_g$ are two bi-neighbourhood functions satisfying the conditions ($\mathrm{C}_{\mathbb{E}}$), ($\mathrm{T}_{\mathbb{E}}$), ($\mathrm{Q}_{\mathbb{C}}$), and ($\mathrm{P}_{\mathbb{C}}$) of Def. 2.1 (but with $\mathcal{N}^{\mathbb{E}}$ and $\mathcal{N}^{\mathbb{C}}$ indexed by g instead of i), and also the following additional conditions:

($\mathrm{F}_{\mathbb{C}}$) $\quad \mathcal{N}^{\mathbb{C}}_{\emptyset}(w) = \emptyset$.

($\mathrm{Int}^2_{\mathbb{EC}}$) $\quad$ If $(\alpha, \beta) \in \mathcal{N}^{\mathbb{E}}_{g_1}(w)$ and $(\gamma, \delta) \in \mathcal{N}^{\mathbb{E}}_{g_2}(w)$, then $(\alpha \cap \gamma, \beta \cup \delta) \in \mathcal{N}^{\mathbb{C}}_{g_1 \cup g_2}(w)$.

The forcing relation $\Vdash$ is defined as in Def. 2.1, in particular:

$\mathcal{M}, w \Vdash \mathbb{E}_g A$ iff there is $(\alpha, \beta) \in \mathcal{N}^{\mathbb{E}}_g(w)$ s.t. for all $v \in \alpha$, $\mathcal{M}, v \Vdash A$, and for all $u \in \beta$, $\mathcal{M}, u \nVdash A$.

$\mathcal{M}, w \Vdash \mathbb{C}_g A$ iff there is $(\alpha, \beta) \in \mathcal{N}^{\mathbb{C}}_g(w)$ s.t. for all $v \in \alpha$, $\mathcal{M}, v \Vdash A$, and for all $u \in \beta$, $\mathcal{M}, u \nVdash A$.

Similarly to logic **ELG** we can prove the following completeness theorem.

Theorem 5.2 *A is derivable in* **COAL** *if and only if it is valid in all bi-neighbourhood models for* **COAL**.

Moreover, by a transformation analogous to the one in Prop. 2.3 we can convert the bi-neighbourhood models for **COAL** into equivalent neighbourhood models for it, as they are defined in [11]: it suffices to assign to the $\mathbb{E}_g$-neighbourhood (resp. the $\mathbb{C}_g$-neighbourhood) of each world w, the subsets γ such that $\alpha \subseteq \gamma \subseteq \mathcal{W} \setminus \beta$ and $(\alpha, \beta) \in \mathcal{N}^{\mathbb{E}}_g(w)$ (resp. $(\alpha, \beta) \in \mathcal{N}^{\mathbb{C}}_g(w)$).

The hypersequent calculus $\mathbf{HS_{COAL}}$ is defined by the propositional rules in Fig. 2 and the modal rules in Fig. 6. As before, each axiom has a corresponding rule in the calculus. An example of derivation is the following.

$$\dfrac{\dfrac{\dfrac{\dfrac{..., \langle A, B\rangle^{\mathbb{C}}_{g_1 \cup g_2} \Rightarrow \mathbb{C}_{g_1 \cup g_2}(A \wedge B) \mid A, B \Rightarrow A \wedge B \qquad ... \mid A \wedge B \Rightarrow A \qquad ... \mid A \wedge B \Rightarrow B}{\mathbb{E}_{g_1} A \wedge \mathbb{E}_{g_2} B, \mathbb{E}_{g_1} A, \mathbb{E}_{g_2} B, \langle A\rangle^{\mathbb{E}}_{g_1}, \langle B\rangle^{\mathbb{E}}_{g_2}, \langle A, B\rangle^{\mathbb{C}}_{g_1 \cup g_2} \Rightarrow \mathbb{C}_{g_1 \cup g_2}(A \wedge B)}\ \mathsf{R}_{\mathbb{C}}}{\mathbb{E}_{g_1} A \wedge \mathbb{E}_{g_2} B, \mathbb{E}_{g_1} A, \mathbb{E}_{g_2} B, \langle A\rangle^{\mathbb{E}}_{g_1}, \langle B\rangle^{\mathbb{E}}_{g_2} \Rightarrow \mathbb{C}_{g_1 \cup g_2}(A \wedge B)}\ \mathsf{Int}^2_{\mathbb{EC}}}{\mathbb{E}_{g_1} A \wedge \mathbb{E}_{g_2} B, \mathbb{E}_{g_1} A, \mathbb{E}_{g_2} B \Rightarrow \mathbb{C}_{g_1 \cup g_2}(A \wedge B)}\ \mathsf{L}_{\mathbb{E}} \times 2}{\mathbb{E}_{g_1} A \wedge \mathbb{E}_{g_2} B \Rightarrow \mathbb{C}_{g_1 \cup g_2}(A \wedge B)}\ \mathsf{L}\wedge$$

By extending the proofs for $\mathbf{HS_{ELG}}$ we can obtain the following theorem.

Theorem 5.3 *All structural rules including* cut *are admissible in* $\mathbf{HS_{COAL}}$. *Moreover,* $\mathbf{HS_{COAL}}$ *is axiomatically complete with respect to* **COAL**, *that is, if A is derivable in* **COAL**, *then $\Rightarrow A$ is derivable in* $\mathbf{HS_{COAL}}$.

Termination of proof search can be obtained by considering a proof search strategy analogous to the one in $\mathbf{HS_{ELG}}$. We only need to consider the following two additional saturation conditions: ($\mathsf{F}_{\mathbb{C}}$) $\langle \Sigma \rangle^{\mathbb{C}}_{\emptyset} \notin \Gamma_n$, and ($\mathsf{Int}^2_{\mathbb{EC}}$) if

$$\mathsf{L}_{\mathbb{E}}\ \frac{G \mid \Gamma, \mathbb{E}_g A, \langle A\rangle^{\mathbb{E}}_g \Rightarrow \Delta}{G \mid \Gamma, \mathbb{E}_g A \Rightarrow \Delta} \qquad \mathsf{L}_{\mathbb{C}}\ \frac{G \mid \Gamma, \mathbb{C}_g A, \langle A\rangle^{\mathbb{C}}_g \Rightarrow \Delta}{G \mid \Gamma, \mathbb{C}_g A \Rightarrow \Delta}$$

$$\mathsf{Int}^1_{\mathbb{E}\mathbb{C}}\ \frac{G \mid \Gamma, \langle \Sigma\rangle^{\mathbb{E}}_g, \langle \Sigma\rangle^{\mathbb{C}}_g \Rightarrow \Delta}{G \mid \Gamma, \langle \Sigma\rangle^{\mathbb{E}}_g \Rightarrow \Delta}$$

$$\mathsf{R}_{\mathbb{E}}\ \frac{G \mid \Gamma, \langle \Sigma\rangle^{\mathbb{E}}_g \Rightarrow \mathbb{E}_g A, \Delta \mid \Sigma \Rightarrow A \qquad \{G \mid \Gamma, \langle \Sigma\rangle^{\mathbb{E}}_g \Rightarrow \mathbb{E}_g A, \Delta \mid A \Rightarrow B\}_{B\in\Sigma}}{G \mid \Gamma, \langle \Sigma\rangle^{\mathbb{E}}_g \Rightarrow \mathbb{E}_g A, \Delta}$$

$$\mathsf{R}_{\mathbb{C}}\ \frac{G \mid \Gamma, \langle \Sigma\rangle^{\mathbb{C}}_g \Rightarrow \mathbb{C}_g A, \Delta \mid \Sigma \Rightarrow A \qquad \{G \mid \Gamma, \langle \Sigma\rangle^{\mathbb{C}}_g \Rightarrow \mathbb{C}_g A, \Delta \mid A \Rightarrow B\}_{B\in\Sigma}}{G \mid \Gamma, \langle \Sigma\rangle^{\mathbb{C}}_g \Rightarrow \mathbb{C}_g A, \Delta}$$

$$\mathsf{C}_{\mathbb{E}}\ \frac{G \mid \Gamma, \langle \Sigma\rangle^{\mathbb{E}}_g, \langle \Pi\rangle^{\mathbb{E}}_g, \langle \Sigma, \Pi\rangle^{\mathbb{E}}_g \Rightarrow \Delta}{G \mid \Gamma, \langle \Sigma\rangle^{\mathbb{E}}_g, \langle \Pi\rangle^{\mathbb{E}}_g \Rightarrow \Delta} \qquad \mathsf{T}_{\mathbb{E}}\ \frac{G \mid \Gamma, \langle \Sigma\rangle^{\mathbb{E}}_g, \Sigma \Rightarrow \Delta}{G \mid \Gamma, \langle \Sigma\rangle^{\mathbb{E}}_g \Rightarrow \Delta}$$

$$\mathsf{Q}_{\mathbb{C}}\ \frac{\{G \mid \Gamma, \langle \Sigma\rangle^{\mathbb{C}}_g \Rightarrow \Delta \mid \Rightarrow B\}_{B\in\Sigma}}{G \mid \Gamma, \langle \Sigma\rangle^{\mathbb{C}}_g \Rightarrow \Delta} \qquad \mathsf{P}_{\mathbb{C}}\ \frac{G \mid \Gamma, \langle \Sigma\rangle^{\mathbb{C}}_g \Rightarrow \Delta \mid \Sigma \Rightarrow}{G \mid \Gamma, \langle \Sigma\rangle^{\mathbb{C}}_g \Rightarrow \Delta}$$

$$\mathsf{F}_{\mathbb{C}}\ \frac{}{G \mid \Gamma, \langle \Sigma\rangle^{\mathbb{C}}_\emptyset \Rightarrow \Delta} \qquad \mathsf{Int}^2_{\mathbb{E}\mathbb{C}}\ \frac{G \mid \Gamma, \langle \Sigma\rangle^{\mathbb{E}}_{g_1}, \langle \Pi\rangle^{\mathbb{E}}_{g_2}, \langle \Sigma, \Pi\rangle^{\mathbb{C}}_{g_1\cup g_2} \Rightarrow \Delta}{G \mid \Gamma, \langle \Sigma\rangle^{\mathbb{E}}_{g_1}, \langle \Pi\rangle^{\mathbb{E}}_{g_2} \Rightarrow \Delta}$$

Fig. 6. Modal rules of $\mathbf{HS_{COAL}}$.

$\langle\Sigma\rangle^{\mathbb{E}}_{g_1}, \langle\Pi\rangle^{\mathbb{E}}_{g_2} \in \Gamma_n$, then $\langle\Omega\rangle^{\mathbb{C}}_{g_1\cup g_2} \in \Gamma_n$ such that $\mathsf{set}(\Omega) = \mathsf{set}(\Sigma,\Pi)$. As for $\mathbf{HS_{ELG}}$ we can prove that proof search always terminates, whence we obtain a decision procedure for the logic **COAL**. Again, proof search is not optimal since derivation can have an exponential size whereas the logic is in PSPACE, as proved by Troquard [11].

We can also prove that the calculus is semantically complete. As before, the proof consists in showing how to extract a countermodel of a non-derivable hypersequent by means of the information provided by the failed proof.

Theorem 5.4 *If H is valid in all bi-neighbourhood models for* **COAL***, then it is derivable in* $\mathbf{HS_{COAL}}$.

Proof. Given a saturated hypersequent H we define a model $\mathcal{M}$ as in Def. 4.3 (replacing agents i with groups g). We can prove that formulas and blocks in the left-hand side of the components are satisfied in the corresponding worlds, and that formulas in the right-hand side are falsified, whence $\mathcal{M}$ is a countermodel of H. Moreover, we can prove that $\mathcal{M}$ is a bi-neighbourhood model for **COAL**. The proofs are as in Lemma 4.4. We only consider the following two conditions.

($\mathrm{Int}^2_{\mathbb{E}\mathbb{C}}$) Assume that $(\alpha,\beta) \in \mathcal{N}^{\mathbb{E}}_{g_1}(n)$ and $(\gamma,\delta) \in \mathcal{N}^{\mathbb{E}}_{g_2}(n)$. If $(\alpha,\beta) \neq (\gamma,\delta)$ or $g_1 \neq g_2$, then there are $\langle\Sigma\rangle^{\mathbb{E}}_{g_1}, \langle\Pi\rangle^{\mathbb{E}}_{g_2} \in \Gamma_n$ such that $\Sigma^+ = \alpha$, $\Sigma^- = \beta$, $\Pi^+ = \gamma$ and $\Pi^- = \delta$. By saturation or rule $\mathsf{Int}^2_{\mathbb{E}\mathbb{C}}$, there is $\langle\Omega\rangle^{\mathbb{C}}_{g_1\cup g_2} \in \Gamma_n$ such that $\mathsf{set}(\Omega) = \mathsf{set}(\Sigma,\Pi)$, thus $(\Omega^+,\Omega^-) \in \mathcal{N}^{\mathbb{C}}_{g_1\cup g_2}(n)$, where, as shown in the proof of Lemma 4.4 case ($\mathrm{C}_{\mathbb{E}}$), $\Omega^+ = \alpha\cap\gamma$ and $\Omega^- = \beta\cup\delta$. If instead $(\alpha,\beta) = (\gamma,\delta)$ and $g_1 = g_2$, then there is $\langle\Sigma\rangle^{\mathbb{E}}_{g_1} \in \Gamma_n$ such that $\Sigma^+ = \alpha$

and $\Sigma^- = \beta$. Then by saturation of rule $\mathsf{Int}_{\mathbb{EC}}$ there is $\langle\Omega\rangle_i^{\mathbb{C}} \in \Gamma_n$ such that $\mathsf{set}(\Sigma) = \mathsf{set}(\Omega)$. Then $(\Omega^+, \Omega^-) = (\Sigma^+, \Sigma^-) = (\alpha, \beta) \in \mathcal{N}_i^{\mathbb{C}}(n)$.

($\mathrm{F}_{\mathbb{C}}$) By saturation of $\mathsf{F}_{\mathbb{C}}$ there is no block $\langle\Sigma\rangle_\emptyset^{\mathbb{C}} \in \Gamma_n$, then $\mathcal{N}_\emptyset^{\mathbb{C}}(n) = \emptyset$. $\square$

We conclude this section by showing that *coalition monotonicity* is not valid in **COAL**. We present the countermodel directly extracted from a failed proof.

Example 5.5 (No coalition monotonicity) We show that the formula $\mathbb{E}_{\{a\}}p \to \mathbb{E}_{\{a,b\}}p$ is not valid in **COAL**. A failed proof is as follows:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\textbf{saturated}}{\langle p\rangle_{\{a\}}^{\mathbb{E}}, \langle p\rangle_{\{a\}}^{\mathbb{C}}, p, \mathbb{E}_{\{a\}}p \Rightarrow \mathbb{E}_{\{a,b\}}p \mid \Rightarrow p}\ \mathsf{Q}_{\mathbb{C}}}{\langle p\rangle_{\{a\}}^{\mathbb{E}}, \langle p\rangle_{\{a\}}^{\mathbb{C}}, p, \mathbb{E}_{\{a\}}p \Rightarrow \mathbb{E}_{\{a,b\}}p}\ \mathsf{Int}_{\mathbb{EC}}}{\langle p\rangle_{\{a\}}^{\mathbb{E}}, p, \mathbb{E}_{\{a\}}p \Rightarrow \mathbb{E}_{\{a,b\}}p}\ \mathsf{T}_{\mathbb{E}}}{\langle p\rangle_{\{a\}}^{\mathbb{E}}, \mathbb{E}_{\{a\}}p \Rightarrow \mathbb{E}_{\{a,b\}}p}\ \mathsf{L}_{\mathbb{E}}}{\mathbb{E}_{\{a\}}p \Rightarrow \mathbb{E}_{\{a,b\}}p}
$$

Let $1 \mapsto \langle p\rangle_a^{\mathbb{E}}, \langle p\rangle_a^{\mathbb{C}}, p, \mathbb{E}_{\{a\}}p \Rightarrow \mathbb{E}_{\{a,b\}}p$, and $2 \mapsto\ \Rightarrow p$. We obtain the following countermodels:

Bi-neighbourhood countermodel: $\mathcal{M} = \langle \mathcal{W}, \mathcal{N}_g^{\mathbb{E}}, \mathcal{N}_g^{\mathbb{C}}, \mathcal{V}\rangle$, where $\mathcal{W} = \{1, 2\}$; $\mathcal{V}(p) = \{1\}$; $\mathcal{N}_{\{a\}}^{\mathbb{E}}(1) = \mathcal{N}_{\{a\}}^{\mathbb{C}}(1) = \{(p^+, p^-)\} = \{(\{1\}, \{2\})\}$; and $\mathcal{N}_g^{\mathbb{E}}(k) = \mathcal{N}_g^{\mathbb{C}}(k) = \emptyset$ for $g \neq \{a\}$ or $k \neq 1$.

Neighbourhood countermodel: $\mathcal{M} = \langle \mathcal{W}, \mathcal{N}_g^{\mathbb{E}}, \mathcal{N}_g^{\mathbb{C}}, \mathcal{V}\rangle$, where $\mathcal{W} = \{1, 2\}$; $\mathcal{V}(p) = \{1\}$; $\mathcal{N}_{\{a\}}^{\mathbb{E}}(1) = \mathcal{N}_{\{a\}}^{\mathbb{C}}(1) = \{\{1\}\}$; and $\mathcal{N}_g^{\mathbb{E}}(k) = \mathcal{N}_g^{\mathbb{C}}(k) = \emptyset$ for $g \neq \{a\}$ or $k \neq 1$.

6 Conclusion

We have presented hypersequent calculi for Elgesem's logic of agency and ability and its coalition extension proposed by Troquard. The calculi have good structural properties, including the syntactical admissibility of cut. Furthermore, we have defined a terminating proof search strategy which ensures that a derivation or a countermodel will be found for every formula. In particular, in case of a failed proof it is possible to directly extract a countermodel of the non-valid formula in the bi-neighbourhood semantics, whence by an easy transformation in the standard neighbourhood semantics. All in all, the calculi provide constructive decision procedures for the two logics.

Troquard has proposed several extensions of his coalition logic with further principles for group agency, such as delegation and strict-joint agency, the latter stating that if a group brings about that A, then any strict subgroup of it cannot bring about that A. We plan to extend our calculi to cover also these extensions, and possibly others. Moreover, our calculi are well-suited for automatisation. We plan to implement them in order to realise the first theorem provers for the logics of agency and ability.

Appendix

Proof of Theorem 3.7. Recall that, for an application of cut, the *cut formula* is the formula which is deleted by that application, while the *cut height* is the sum of the heights of the derivations of the premisses of cut. We prove that: **(A)** $\forall c.Cut(c,0)$. **(B)** $\forall h.Cut(0,h)$. **(C)** $\forall c.(\forall h.Cut(c,h) \to Sub(c))$. **(D)** $\forall c.\forall h.((\forall c' < c.(Sub(c') \land \forall h'.Cut(c',h')) \land \forall h'' < h.Cut(c,h'')) \to Cut(c,h))$.

(A) and **(B)** are trivial. **(C)** Assume $\forall h Cut(c,h)$. The proof is by induction on the height m of the derivation of $G \mid \overrightarrow{\langle A^n,\Pi\rangle_i^{\mathbb{E}}}, \overrightarrow{\langle A^m,\Omega\rangle_j^{\mathbb{C}}}, \Gamma \Rightarrow \Delta$. We only consider the case where $m > 0$ and at least one block among $\overrightarrow{\langle A^n,\Pi\rangle_i^{\mathbb{E}}}, \overrightarrow{\langle A^m,\Omega\rangle_j^{\mathbb{C}}}$ is principal in the last rule application. We consider as an example the case where the last rule applied is $\mathsf{Int}_{\mathbb{EC}}$:

$$\dfrac{G \mid \langle A^{n_k},\Pi_k\rangle_i^{\mathbb{E}}, \langle A^{n_k},\Pi_k\rangle_i^{\mathbb{C}}, \Gamma \Rightarrow \Delta}{G \mid \langle A^{n_k},\Pi_k\rangle_i^{\mathbb{E}}, \Gamma \Rightarrow \Delta}\ \mathsf{Int}_{\mathbb{EC}}$$

By applying the inductive hypothesis to the premiss we obtain $G \mid \langle \Sigma^{n_k},\Pi_k\rangle_i^{\mathbb{E}}, \langle \Sigma^{n_k},\Pi_k\rangle_i^{\mathbb{C}}, \Gamma \Rightarrow \Delta$, then by $\mathsf{Int}_{\mathbb{EC}}$ we derive $G \mid \langle \Sigma^{n_k},\Pi_k\rangle_i^{\mathbb{C}}, \Gamma \Rightarrow \Delta$.

(D) Assume $\forall c' < c.\,(Sub(c') \land \forall h'.\,Cut(c',h'))$ and $\forall h'' < h.\,Cut(c,h'')$. We show that all applications of cut of height h on a cut formula of weight c can be replaced by different applications of cut, either of smaller height or on a cut formula of smaller weight. We only consider the cases where $h, c > 0$ and the cut formula is $\mathbb{E}_i B$, principal in the derivation of both premisses of cut:

$$\dfrac{\mathsf{R}_{\mathbb{E}}\ \dfrac{\begin{matrix} G \mid \langle\Sigma\rangle_i^{\mathbb{E}}, \Gamma \Rightarrow \Delta, \mathbb{E}_i B \mid \Sigma \Rightarrow B \\ \vdots \end{matrix} \quad \{G \mid \langle\Sigma\rangle_i^{\mathbb{E}}, \Gamma \Rightarrow \Delta, \mathbb{E}_i B \mid B \Rightarrow C\}_{C\in\Sigma}}{G \mid \langle\Sigma\rangle_i^{\mathbb{E}}, \Gamma \Rightarrow \Delta, \mathbb{E}_i B} \qquad \dfrac{G \mid \langle B\rangle, \mathbb{E}_i B, \langle\Sigma\rangle_i^{\mathbb{E}}, \Gamma \Rightarrow \Delta}{G \mid \mathbb{E}_i B, \langle\Sigma\rangle_i^{\mathbb{E}}, \Gamma \Rightarrow \Delta}\ \mathsf{L}_{\mathbb{E}}}{G \mid \langle\Sigma\rangle_i^{\mathbb{E}}, \Gamma \Rightarrow \Delta}\ \mathsf{cut}$$

The derivation is converted as follows, with several applications of cut of smaller height and an admissible application of sub.

$$\dfrac{G \mid \langle\Sigma\rangle_i^{\mathbb{E}}, \Gamma \Rightarrow \Delta, \mathbb{E}_i B \mid \Sigma \Rightarrow B \qquad \dfrac{G \mid \mathbb{E}_i B, \langle\Sigma\rangle_i^{\mathbb{E}}, \Gamma \Rightarrow \Delta}{G \mid \mathbb{E}_i B, \langle\Sigma\rangle_i^{\mathbb{E}}, \Gamma \Rightarrow \Delta \mid \Sigma \Rightarrow B}\ \mathsf{Ewk}}{① \ G \mid \langle\Sigma\rangle_i^{\mathbb{E}}, \Gamma \Rightarrow \Delta \mid \Sigma \Rightarrow B}\ \mathsf{cut}$$

$$\dfrac{\dfrac{\mathsf{Lwk}\ \dfrac{G \mid \langle\Sigma\rangle_i^{\mathbb{E}}, \Gamma \Rightarrow \Delta, \mathbb{E}_i B}{G \mid \langle B\rangle_i^{\mathbb{E}}, \langle\Sigma\rangle_i^{\mathbb{E}}, \Gamma \Rightarrow \Delta, \mathbb{E}_i B} \qquad G \mid \langle B\rangle, \mathbb{E}_i B, \langle\Sigma\rangle_i^{\mathbb{E}}, \Gamma \Rightarrow \Delta}{G \mid \langle B\rangle_i^{\mathbb{E}}, \langle\Sigma\rangle_i^{\mathbb{E}}, \Gamma \Rightarrow \Delta}\ \mathsf{cut}}{② \ G \mid \langle\Sigma\rangle_i^{\mathbb{E}}, \Gamma \Rightarrow \Delta \mid \langle B\rangle_i^{\mathbb{E}}, \langle\Sigma\rangle_i^{\mathbb{E}}, \Gamma \Rightarrow \Delta}\ \mathsf{Ewk}$$

$$\dfrac{\dfrac{\dfrac{① \qquad \left\{ \dfrac{G \mid \langle\Sigma\rangle_i^{\mathbb{E}}, \Gamma \Rightarrow \Delta, \mathbb{E}_i B \mid B \Rightarrow C \qquad \dfrac{G \mid \mathbb{E}_i B, \langle\Sigma\rangle_i^{\mathbb{E}}, \Gamma \Rightarrow \Delta}{G \mid \mathbb{E}_i B, \langle\Sigma\rangle_i^{\mathbb{E}}, \Gamma \Rightarrow \Delta \mid B \Rightarrow C}\ \mathsf{Ewk}}{G \mid \langle\Sigma\rangle_i^{\mathbb{E}}, \Gamma \Rightarrow \Delta \mid B \Rightarrow C}\ \mathsf{cut} \right\}_{C\in\Sigma} \qquad ②}{G \mid \langle\Sigma\rangle_i^{\mathbb{E}}, \Gamma \Rightarrow \Delta \mid \langle\Sigma\rangle_i^{\mathbb{E}}, \langle\Sigma\rangle_i^{\mathbb{E}}, \Gamma \Rightarrow \Delta}\ \mathsf{sub}}{G \mid \langle\Sigma\rangle_i^{\mathbb{E}}, \Gamma \Rightarrow \Delta \mid \langle\Sigma\rangle_i^{\mathbb{E}}, \Gamma \Rightarrow \Delta}\ \mathsf{Lctr}}{G \mid \langle\Sigma\rangle_i^{\mathbb{E}}, \Gamma \Rightarrow \Delta}\ \mathsf{Ectr}$$

□

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